

PENROSE LIMITS VS STRING EXPANSIONS

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We analyse the relation between two a priori quite different expansions of the string equations of motion and constraints in a general curved background, namely one based on the covariant Penrose-Fermi expansion of the metric $G_{\mu\nu}$ around a Penrose limit plane wave associated to a null geodesic γ , and the other on the Riemann coordinate expansion in the exact metric $G_{\mu\nu}$ of the string embedding variables around the null geodesic γ . Starting with the observation that there is a formal analogy between the exact string equations in a plane wave and the first order string equations in a general background, we show that this analogy becomes exact provided that one chooses the background string configuration to be the null geodesic γ itself. We then explore the higher-order correspondence between these two expansions and find that for a general curved background they agree to all orders provided that one works in Fermi coordinates and in the lightcone gauge. Requiring moreover the conformal gauge restricts one to the usual class of (Brinkmann) backgrounds admitting simultaneously the lightcone and the conformal gauge, without further restrictions.

1 INTRODUCTION

After the initial developments [1, 2, 3] related to the discovery of the maximally supersymmetric IIB plane wave and its connection with the Penrose limit [4, 5], much effort has, in the wake of the seminal BMN paper [6], understandably gone into exploring the consequences of these ideas in the context of the AdS/CFT correspondence, eventually leading to deep new insights into the integrable structures underlying the theories on both sides of the correspondence. Some of these developments are described e.g. in [7, 8, 9].

Along a different line, in a series of papers [10, 11, 12, 13, 14, 15] we have explored various aspects of the geometry and physics of plane waves and Penrose limits *per se*, also with the expectation that these results will eventually lead to further insights into the gauge theory – geometry correspondence. In particular, in [13, 14] we provided a geometrically transparent and covariant characterisation of the Penrose limit map $(G_{\mu\nu}, \gamma) \mapsto A_{ab}$ that associates to a space-time metric $G_{\mu\nu}$ and a null geodesic γ the wave profile A_{ab} characterising the Penrose limit plane wave metric $ds^2 = 2dx^+dx^- + A_{ab}(x^+)x^ax^b(dx^+)^2 + \delta_{ab}dx^adx^b$. Namely, the $A_{ab}(x^+) = -R_{+a+b}(x^+)$, which are the only non-vanishing coordinate components of the curvature tensor of the plane wave, are at the same time simply certain frame components of the curvature tensor of the original metric $G_{\mu\nu}$, restricted to the null geodesic γ (with affine parameter x^+) along which the Penrose limit is taken.

In [15], we used Fermi coordinates based on the null geodesic γ to generalise the above result to an all order covariant expansion of a metric around its Penrose limit (covariant in the sense that all the higher order terms are also expressed in terms of the Riemann tensor of the original metric and its derivatives). In the following we will refer to this expansion as the *Penrose-Fermi expansion* of a metric.

Within this clear geometric setting it is now possible to address questions regarding the relation between the dynamics of various objects in the original metric and its Penrose limit. In particular, the above geometric interpretation of the Penrose limit can be re-interpreted as providing an answer to the

QUESTION: What is the interpretation of the geodesic equation in the Penrose limit plane wave (associated to the metric $G_{\mu\nu}$ and a null geodesic γ) in terms of the original data $(G_{\mu\nu}, \gamma)$?

ANSWER: It is simply the transverse geodesic deviation equation for $(G_{\mu\nu}, \gamma)$.

It is then natural to next ask the same question for strings rather than for particles.¹

QUESTION: What is the interpretation of the string equations of motion in the Penrose limit plane wave in terms of the string equations of motion in the original metric $G_{\mu\nu}$?

Thinking about this, one quickly realises that this will have to be related to a (first order) expansion of the string embedding variables $X^\mu(\tau, \sigma)$ around the null geodesic $\gamma(\tau)$, the latter regarded as a string background solution of the equations of motion and constraints in the original space-time with metric $G_{\mu\nu}$.

Thus, in more general terms what this amounts to is a comparison of two apparently quite different expansions of the string equations in a curved background, an expansion of the metric itself (the Penrose-Fermi expansion of $G_{\mu\nu}$) on the one hand, and an expansion of the string embedding variables around a background string configuration (but in the exact metric $G_{\mu\nu}$) on the other.

In order to be able to assess what the advantages (or perhaps drawbacks) are of choosing a null geodesic as a (somewhat degenerate) string background configuration, we have found it useful to begin the discussion with an analysis of the expansion of the string equations around a non-degenerate string background configuration $X_B^\mu(\tau, \sigma)$. This is, of course, largely classical material, the Riemann coordinate expansion of the two-dimensional sigma-model having been discussed at length e.g. in [17], and we briefly recall this (and adapt it to the present setting) in section 2 and appendix A.1.

The principal difference to the discussion of [17] is that, in addition to the sigma-model equations of motion we also have to deal with the string constraints. Then the main observation of this section is that, to first order in an expansion around a classical string configuration $X_B(\tau, \sigma)$ in an arbitrary curved background, these constraints allow one to explicitly solve for the tangential fluctuations and to completely eliminate them from the equations of motion for the true dynamical transverse degrees of freedom. While the result as such may not be surprising (it is

¹In a similar spirit, in [16] we showed that scalar field probes of space-time singularities exhibit a universal behaviour that is strictly analogous to that of massless particle probes (i.e. the Penrose limit) uncovered in [13, 14].

essentially a consequence of worldsheet diffeomorphism invariance), our presentation is aimed at highlighting the analogies and differences with strings in the conformal and lightcone gauge in plane wave backgrounds.

We pursue this analogy in section 3, where we observe first of all that the main difference between the first order and plane wave equations of motion for the true dynamical transverse degrees of freedom is due to the extrinsic curvature of the background string X_B . We then argue that this difference disappears, and that the analogy becomes perfect, when one chooses the background string configuration to be a null geodesic, $X_B(\tau, \sigma) \rightarrow \gamma(\tau)$. The result of this section can then be summarised as the answer to the question posed above.

ANSWER: The exact transverse string equations in the first order Penrose-Fermi expansion of the metric $G_{\mu\nu}$ around γ , i.e. in the Penrose limit plane wave metric associated to $G_{\mu\nu}$ and γ , are equivalent to the transverse first-order string expansion equations around a null geodesic γ in the original background $G_{\mu\nu}$.

Finally, in section 4 we address the

QUESTION: What can one say about the correspondence between the string expansion on the one hand and the Penrose-Fermi expansion on the other, established to first order in section 3, at higher orders?

This boils down to a comparison of two different prescriptions for how to describe the locus of nearby strings in terms of geodesic distance (namely via Riemann or Fermi coordinates). We show that demanding all order equivalence of the two expansions is tantamount to the requirement that the string be comoving with the null geodesic, and these geometric considerations then lead to the

ANSWER: Provided that one works in Fermi coordinates and in the lightcone gauge, these two expansions agree to all orders.

This combined lightcone (worldsheet) and Fermi (space-time) gauge (i.e. writing the metric in Fermi coordinates) is, a priori, always available. Frequently, however, the lightcone gauge is imposed in conjunction with the conformal gauge, and this requires a metric that has a parallel null vector, as well as a coordinate system in which this is a coordinate vector ∂_v [21]. We show (appendix B) that for all such metrics the latter requirement is actually compatible with the Fermi gauge. Since for this class of metrics canonical quantisation becomes particularly tractable in the lightcone and conformal gauge, this makes this all order equivalence especially appealing.

These results provide us with what seems to be a satisfactory overall geometric picture of the relation between string dynamics in a general curved background and in the Penrose-Fermi expansion of that background around its Penrose limit plane wave metric.

We should also note here in passing that the idea of basing a string expansion on an expansion around a geodesic is as such of course not new. Such an expansion was e.g. considered (to first order) in [19], primarily for specific examples of metric backgrounds, and using (for reasons we do not fully comprehend) timelike instead of null geodesics. An expansion based on null geodesics was considered in [20], in the context of tensionless strings. While formally similar,

our treatment of this expansion is quite different, both technically (using in an essential way the manifestly covariant Riemann and Fermi coordinate expansions) and in spirit. E.g. we argued in [3, 10] that the Penrose limit is most naturally understood as a particular large tension $\alpha' \rightarrow 0$ limit, and in the present context the Riemann coordinate (derivative) expansion we employ can, as usual, be translated into an α' expansion.

2 COVARIANT STRING EXPANSION AROUND A REGULAR STRING BACKGROUND SOLUTION

Our point of departure is the Polyakov action

$$S[X, h] = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ij} G_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu, \quad (2.1)$$

for a string moving in the D -dimensional curved space-time background described by the metric $G_{\mu\nu}$, with $X^\mu = X^\mu(\tau, \sigma)$ the string embedding variables corresponding to the target space coordinates x^μ , and h_{ij} the worldsheet metric. Throughout this paper, with the exception of section 4, we work in the conformal gauge $h_{ij} = e^\phi \eta_{ij}$, leading to the sigma-model action (the conformal factor e^ϕ drops out of all subsequent equations)

$$S[X] = \frac{1}{2\pi\alpha'} \int d^2\sigma G_{\mu\nu}(X) \partial^i X^\mu \partial_i X^\nu. \quad (2.2)$$

The equations of motion (e.o.m.)

$$\nabla^i \partial_i X^\mu = \partial^i \partial_i X^\mu + \Gamma_{\nu\lambda}^\mu(X) \partial^i X^\nu \partial_i X^\lambda = 0 \quad (2.3)$$

have to be supplemented by the constraints

$$G_{\mu\nu}(X) \partial_\pm X^\mu \partial_\pm X^\nu = 0, \quad (2.4)$$

written here in worldsheet lightcone coordinates $\sigma^\pm = (\sigma \pm \tau)/\sqrt{2}$.

We will now expand the action covariantly around a background string solution X_B^μ of (2.3). The standard technique for this is the Riemann coordinate expansion $X^\mu = X_B^\mu + \xi^\mu$ discussed in detail in the sigma-model context in [17] and briefly recalled in appendix A.

For the time being, in order to compare the Riemann coordinate expansion with the Penrose limit, we are only interested in the lowest non-trivial order of this expansion. The e.o.m. for the expansion fields ξ^μ (most readily obtained by expanding and then varying the action) are

$$\nabla_i \nabla^i \xi^\lambda + R_{\mu\rho_1\nu}^\lambda \partial_i X_B^\mu \partial^i X_B^\nu \xi^{\rho_1} = 0. \quad (2.5)$$

The corresponding first-order constraints are calculated by expanding (2.4) accordingly (A.9), and read

$$G_{\mu\nu} \nabla_\pm \xi^\mu \partial_\pm X_B^\nu = 0. \quad (2.6)$$

It is now convenient to introduce a frame $E_A^\mu(X_B)$ along the worldsheet. The components tangential to the worldsheet E_i^μ , $i \in \{+, -\}$ or $\{\tau, \sigma\}$, are chosen to be the derivatives along the

coordinate lines of the conformal gauge coordinate system, viewed as the stringy generalisation of the geodesic affine parameter, i.e.

$$E_i = \partial_i, \quad E_i^\mu = \partial_i X_B^\mu, \quad (2.7)$$

completed by an orthonormal frame E_a^μ , $a \in \{2, \dots, D-1\}$ (determined up to transverse orthogonal frame rotations), such that

$$G_{\mu\nu} E_i^\mu E_j^\nu = g_{ij}, \quad G_{\mu\nu} E_i^\mu E_a^\nu = 0, \quad G_{\mu\nu} E_a^\mu E_b^\nu = \delta_{ab}. \quad (2.8)$$

Thus g_{ij} is the induced metric on the classical worldsheet background (constrained to be conformally flat by the conformal gauge condition). The string e.o.m (2.3) can now simply be written as $\nabla^i E_i = 0$, replacing the auto-parallelity condition $\nabla_\tau E_\tau = 0$ of a geodesic. They can be supplemented by the integrability conditions $\epsilon^{ij} \nabla_i E_j = 0$, which are due to the fact that the E_i are coordinate vectors. In terms of the worldsheet lightcone coordinates σ^\pm , these two equations can then be written in the condensed and useful form

$$\nabla_\pm E_\mp = 0. \quad (2.9)$$

After decomposition of the expansion fields into their tangential and transverse components,

$$\xi^\mu = \xi^A E_A^\mu = \xi^i E_i^\mu + \xi^a E_a^\mu, \quad (2.10)$$

one can reformulate the action, e.o.m. and the constraints in frame component form. Using (2.8) and (2.9), we find for the latter

$$g_{+-} \partial_\pm \xi^\mp - G_{\mu\nu} E_a^\mu \nabla_\pm E_\pm^\nu \xi^a = 0. \quad (2.11)$$

These constraints can be solved for the (tangential, longitudinal) lightcone components ξ^\pm , up to the residual gauge freedom $\xi^\pm \rightarrow \xi^\pm + f^\pm(\sigma^\pm)$. Therefore their e.o.m. must be satisfied identically by virtue of the constraints. Indeed, after a lengthy calculation we find that the tangential components of (2.5) are just the derivatives of (2.11), i.e.

$$\partial_\mp (g_{+-} \partial_\pm \xi^\mp - G_{\mu\nu} E_a^\mu \nabla_\pm E_\pm^\nu \xi^a) = 0. \quad (2.12)$$

Furthermore, since the tangential components ξ^\pm appear in the transverse components of the e.o.m. (2.5)

$$\begin{aligned} & \partial_+ \partial_- \xi^a + G_{\mu\nu} (E^{a\mu} \nabla_+ E_+^\nu \partial_- \xi^+ + E^{a\mu} \nabla_+ E_b^\nu \partial_- \xi^b + E^{a\mu} \nabla_- E_-^\nu \partial_+ \xi^- + E^{a\mu} \nabla_- E_b^\nu \partial_+ \xi^b) \\ & + \frac{1}{2} G_{\mu\nu} (E^{a\mu} \nabla_+ \nabla_- E_b^\nu \xi^b + E^{a\mu} \nabla_- \nabla_+ E_b^\nu \xi^b) + \frac{1}{2} R^a_{+b-} \xi^b + \frac{1}{2} R^a_{-b+} \xi^b = 0 \end{aligned} \quad (2.13)$$

only via their derivatives $\partial_\pm \xi^\mp$, we can use the constraints (2.11) to completely eliminate them. One then finds the purely transverse e.o.m.

$$\begin{aligned} & \left(\frac{1}{2} \partial^i \partial_i \delta_b^a + G_{\mu\nu} E^{a\mu} \nabla^i E_b^\nu \partial_i + G_{\mu\nu} \nabla^i E^{a\mu} \nabla_i E_b^\nu - (G_{\mu\nu} \nabla^i E^{a\mu} E_c^\nu) (G_{\lambda\kappa} E^{c\lambda} \nabla_i E_b^\kappa) \right. \\ & \left. + \frac{1}{2} G_{\mu\nu} E^{a\mu} \nabla^i \nabla_i E_b^\nu + \frac{1}{2} R^{ai}_{bi} \right) \xi^b = 0. \end{aligned} \quad (2.14)$$

Thus we have shown that, to first order in an expansion around a classical string configuration X_B in an arbitrary curved background, the tangential fluctuations can be explicitly solved for and eliminated from the e.o.m. for the true dynamical transverse degrees of freedom by virtue of the constraints (2.11).

We conclude this section with two comments on these observations:

1. First of all, the fact that the tangential components ξ^\pm can, in principle, be eliminated to first order is of course related to the underlying worldsheet diffeomorphism invariance. The crucial point here is that (2.11) shows how they can explicitly, and thus in practice, be eliminated in the already partially gauge fixed (conformal gauge) theory. This should be contrasted with the world-sheet covariant approach, e.g. based on the Nambu-Goto action, in which the tangential components, identified to first order with generators of worldsheet diffeomorphisms, can be set to zero (or drop out of the equations) by virtue of the worldsheet diffeomorphism invariance (for a geometrically transparent discussion of these issues see e.g. [23, 24]). However, this is no longer possible (or true) at higher orders in the expansion, which, in contrast to the first order, encode information beyond mere infinitesimal deviations of nearby strings, and thus are not (to the same extent) susceptible to worldsheet diffeomorphisms. Thus if one wants to go to higher orders (as we will eventually do in section 4), the simplest way to control the world-sheet diffeomorphisms is to start with a gauge fixed action and to then simply expand it together with the constraints, exactly as we have done here to first order.
2. Secondly, this is evidently quite reminiscent of the standard treatment of strings in the lightcone gauge, available for plane wave (or more general pp-wave or Brinkmann metric) backgrounds [21]. We will pursue this analogy in the subsequent section. To that end it will be useful to rewrite (2.14) in a manner that makes the underlying geometric structure more manifest, by introducing the gauge covariant derivative w.r.t. transverse frame rotations D_i and the extrinsic curvature of the world-sheet K_{ij}^a ,

$$D_i \xi^a = \partial_i \xi^a + G_{\mu\nu} E^{a\mu} \nabla_i E_b^\nu \xi^b, \quad K_{ij}^a = G_{\mu\nu} E_i^\mu \nabla_j E^{\nu a}. \quad (2.15)$$

In terms of these, (2.14) can be written more transparently as (see e.g. [23])

$$g^{ij} D_i D_j \xi^a + g^{ij} g^{kl} K_{ik}^a K_{jl} \xi^b + g^{ij} R_{jbi}^a \xi^b = 0. \quad (2.16)$$

3 TRANSITION FROM STRINGS TO NULL GEODESICS AS BACKGROUND FIELDS

As mentioned above, the explicit elimination of the lightcone degrees of freedom ξ^\pm from the first order string expansion by virtue of the constraints is strikingly reminiscent of the string e.o.m. in a Penrose limit expansion of the metric whose first order is the plane wave metric

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = 2dx^+ dx^- - R_{+a+b}(x^+) x^a x^b dx^+ dx^+ + \delta_{ab} dx^a dx^b. \quad (3.1)$$

Imposing the conformal gauge, the e.o.m. for $X^+(\tau, \sigma)$ is just the free wave equation

$$(\partial_\tau^2 - \partial_\sigma^2) X^+ = 0, \quad (3.2)$$

and one can fix the residual worldsheet diffeomorphism invariance by choosing the lightcone gauge $X^+(\tau, \sigma) = \tau$. In this gauge, X^- is determined by the constraints

$$\begin{aligned} \dot{X}^- - \frac{1}{2}R_{a+b+}X^aX^b + \frac{1}{2}\delta_{ab}(\dot{X}^a\dot{X}^b + X^{a'}X^{b'}) &= 0 \\ X^{-'} + \delta_{ab}\dot{X}^aX^{b'} &= 0, \end{aligned} \quad (3.3)$$

and its e.o.m.

$$(\partial_\tau^2 - \partial_\sigma^2)X^- + 2R_{+ab+}\partial_\tau X^aX^b + \frac{1}{2}\partial_+R_{+ab+}X^aX^b = 0 \quad (3.4)$$

is then, as in section 2, identically satisfied by virtue of the constraints. The e.o.m. for the remaining transverse variables X^a are simply

$$(-\partial_\tau^2 + \partial_\sigma^2)X^a - R^a_{+b+}(\tau)X^b = 0. \quad (3.5)$$

Now these equations are quite similar to the transverse equations of motion (2.14, 2.16), the difference between the two being mainly due to the complicated extrinsic curvature information of the background string X_B encoded in the second term of (2.16).

Thinking of strings as probes of the background geometry, one is tempted to say that the complicated (extrinsic) geometry of the probe itself obscures or contaminates the background geometry. This becomes most obvious in flat space where the first order string expansion equations about an excited string look much more complicated than the exact string equations themselves. On top of that, for generic curved backgrounds it is typically very hard to find even one exact solution X_B of the non-linear string e.o.m.

It is thus legitimate to ask if there is not a better way to perform a string expansion, one which rids us of all the (for present purposes largely superfluous) geometric information encoded in the extrinsic geometry of the string. Of course the first guess is to try a simpler background X_B , ideally an object with vanishing extrinsic curvature satisfying the exact string e.o.m. and constraints. All of these conditions are satisfied by choosing $X_B(\tau, \sigma) = \gamma(\tau)$ to be a null geodesic since

- for $X_B(\tau, \sigma) = \gamma(\tau)$, the e.o.m. (2.3) reduce to the geodesic equation;
- the constraints (2.4) reduce to the condition that this geodesic be null;
- the extrinsic curvature (2.15) of $\gamma(\tau)$ vanishes, since a geodesic extremises proper time.

The validity of the first two statements is obvious. As regards the third claim, note that in general an extremal submanifold is characterised by the vanishing of the trace of the extrinsic curvature. For a one-dimensional object this is equivalent to vanishing of the extrinsic curvature itself, the condition $K^a_{\tau\tau} = 0$ being just another way of writing the geodesic equation.

As we will see in the following, this choice of background will remedy all the shortcomings mentioned above and, in the end, lead to a first order string expansion equation of the form (3.5).

First of all we need to address the issue how to formulate the string expansion around this somewhat degenerate (because σ -independent) string background $X_B(\tau, \sigma) = \gamma(\tau)$. It turns out

that simply making the replacement $X_B^\mu \rightarrow \gamma^\mu$, while retaining the τ and σ -dependence of ξ , so that e.g.

$$\partial_\tau X_B^\mu = \dot{\gamma}^\mu \quad \partial_\sigma X_B^\mu = \partial_\sigma \gamma^\mu = 0 \quad \nabla_\sigma \xi^\mu = \partial_\sigma \xi^\mu, \quad (3.6)$$

yields valid expansions of the action, constraints and the e.o.m. Therefore we get from (2.5) the e.o.m.

$$(-\nabla_\tau^2 + \partial_\sigma^2)\xi^\lambda - R^\lambda_{\mu\rho_1\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu\xi^{\rho_1} = 0 \quad (3.7)$$

while the constraints (2.6) reduce to

$$G_{\mu\nu}\nabla_\tau\xi^\mu\dot{\gamma}^\nu = 0 \quad G_{\mu\nu}\partial_\sigma\xi^\mu\dot{\gamma}^\nu = 0. \quad (3.8)$$

Using the geodesic equation of motion, these constraints can be integrated to $G_{\mu\nu}\xi^\mu\dot{\gamma}^\nu = c$ with some constant c . We will now show that this constant can be set to zero. Assume a general solution $\xi(\tau, \sigma)$ of the e.o.m. (3.7) and the constraints (3.8), and consider the shifted expansion vector $\tilde{\xi}(\tau, \sigma) = \xi(\tau, \sigma) - c\xi_0(\tau)$, where $\xi_0(\tau)$ satisfies the ordinary geodesic deviation equation with respect to γ , and is normalised according to $G_{\mu\nu}\xi_0^\mu\dot{\gamma}^\nu = 1$. Then $\tilde{\xi}^\mu$ still satisfies the e.o.m. (3.7), but the constraint is

$$G_{\mu\nu}\tilde{\xi}^\mu\dot{\gamma}^\nu = 0. \quad (3.9)$$

In the following we consider two solutions of the first order string expansion to be equivalent if they differ only by a solution of the mere geodesic deviation equation, corresponding essentially just to a rigid displacement of the background geodesic, and consistently set $c = 0$.

Further simplifications arise after introduction of a parallel transported quasi-orthonormal frame E_μ^A (with $E_+^\mu = \dot{\gamma}^\mu$) along the null geodesic γ , as in (A.10), since one then has, expanding $\xi^\mu = \xi^A E_A^\mu$ in this basis, $\nabla_\tau \xi^\mu = (\partial_\tau \xi^A) E_A^\mu$, so that all covariant derivatives can be replaced by partial derivatives acting on the frame components. Hence in frame components the e.o.m. (3.7) are simply

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^A + R^A_{+B+}\xi^B = 0, \quad (3.10)$$

while the choice $c = 0$ (3.9) is tantamount to $\xi^-(\tau, \sigma) = 0$. This condition is strictly analogous to the standard condition one imposes in the construction of the transverse geodesic deviation matrix [25] ($Z^- = 0$ in the notation of [14, section 2.1]). Thus, for the individual frame components one finds

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^+ = -R^+_{+B+}\xi^B = -R^+_{+-+}\xi^- - R^+_{+a+}\xi^a = -R^+_{+a+}\xi^a \quad (3.11)$$

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^- = -R^-_{+B+}\xi^B \equiv 0 \quad (3.12)$$

$$(\partial_\tau^2 - \partial_\sigma^2)\xi^a = -R^a_{+B+}\xi^B = -R^a_{+-+}\xi^- - R^a_{+b+}\xi^b = -R^a_{+b+}\xi^b. \quad (3.13)$$

In particular, the transverse equations (3.13) are now identical to the exact transverse string equations (3.5) in a plane wave background. As regards the equation for ξ^- , on the other hand, comparison with the exact equation (3.4) shows that $\xi^- = 0$ is only a solution to the e.o.m. to lowest order in the Riemann expansion - consistent with the fact that in the scaling (A.17) leading to the Penrose plane wave limit X^- is treated as higher order relative to the X^a .

We conclude that the exact transverse string equations in the first order Penrose-Fermi expansion of the metric $G_{\mu\nu}$ around γ , i.e. in the Penrose limit plane wave metric associated to $G_{\mu\nu}$ and γ , are equivalent to the transverse first-order string equations obtained by expanding the string embedding functions around a null geodesic γ in the original background $G_{\mu\nu}$.

4 THE CORRESPONDENCE TO ALL ORDERS

To what degree and for which metric/geodesic backgrounds can we expect the correspondence between the string expansion and the Penrose-Fermi expansion, which we established above to first order, to be valid at higher orders? To answer this question it is worthwhile to take a step back and compare the geometric set-up in both cases. Although the underlying interpretation is that of an expansion of the embedding variables on the one hand, and of the metric on the other, in the end it all reduces to a different prescription for how to describe the locus of nearby strings in terms of geodesic distance. This is mirrored by the different adapted coordinate systems used, i.e. Riemann vs. Fermi coordinates.

The Riemann coordinates ξ^+ , ξ^- and ξ^a , used as the embedding variables in the string expansion, describe the instantaneous distance to a lightlike particle $\gamma(\tau)$. The somewhat awkward feature of this coordinate system (in the present context) is that, as this particle moves along γ , these coordinates changes (differentiably) with the affine parameter, i.e. with time.

The Penrose-Fermi expansion, on the other hand, is based on Fermi coordinates x^+ , x^- and x^a adapted to the null geodesic γ [15]. In Fermi coordinates, one measures distance w.r.t. the null geodesic as a one-dimensional object. To this end space-time is foliated into transverse hypersurfaces which are parametrised by the affine parameter, promoted to the Fermi coordinate $x^+ = \tau$, and covered with $D - 1$ dimensional, time-independent Riemann coordinates x^- and x^a around the intersection point of geodesic and hypersurface.

At a given but fixed time $\tau = \tau_0$, the position of the string is described by

$$X^\mu(\tau_0, \sigma) = \gamma^\mu(\tau_0) + \Delta X^\mu(\xi((\tau_0, \sigma))), \quad (4.1)$$

and generically $\xi^\mu(\tau_0, \sigma)$ will not lie in the corresponding transverse hypersurface, because the string is not comoving with the null geodesic. In that case, the first construction (Riemann coordinates), in which one simply has $X^\mu(\tau_0, \sigma) = \gamma^\mu(\tau_0) + \xi^\mu(\tau_0, \sigma)$ (A.4), is more convenient and efficient than the Fermi construction, as it accounts for the free movement of the string in space-time.

However, this discussion also shows that both approaches should agree completely if the string is actually confined to comove with the null geodesic. To make this more precise, note that comovement in terms of Fermi coordinates is equivalent to

$$X^+(\tau, \sigma) = \tau, \quad (4.2)$$

i.e. precisely the lightcone gauge condition, whereas in the Riemann string expansion it simply means

$$\xi^+(\tau, \sigma) = 0. \quad (4.3)$$

Now, by construction the transverse Fermi coordinates $(x^{\bar{a}}) = (x^-, x^a)$ are equal to the remaining transverse Riemann coordinates (A.11),

$$x^{\bar{a}} = \xi_{\gamma(\tau)}^{\bar{a}}. \quad (4.4)$$

Thus for comoving strings (lightcone gauge), the two prescriptions to measure the locus of the string, namely transverse distance from the geodesic, indeed agree. In that special case it is enlightening to recalculate the manifest covariant form of the string expansion using Fermi and not Riemann coordinates. As we will show, this significantly simplifies the identification of the tensorial structures at intermediate steps of the calculation, and demonstrates that Fermi coordinates are the ideal reference system to describe the perturbative string expansion in the lightcone gauge.

To see this, recall first that in Riemann coordinates one has the simple relationship $X^\mu(\gamma, \xi) = \gamma^\mu + \xi^\mu$ for the embedding variable, while the expression for its τ -derivative is more complicated (essentially because Riemann coordinates are anchored at a fixed basepoint and thus change as one moves along γ) and given by the infinite series (A.6, A.7).

In Fermi coordinates, on the other hand, the initial expression for the expansion of $X^A(\gamma, \xi)$ is somewhat more complicated, being given by the infinite series (A.15), but since this expression holds along the entire null geodesic, no new terms are generated when taking the τ -derivative (A.16). The simple (but crucial) observation is now that, upon using (4.3), this expansion (A.15) collapses to the simple result

$$X^A(\gamma, \xi) = \delta_+^A \tau + \delta_a^A \xi^{\bar{a}}, \quad (4.5)$$

in accordance with (4.2) and (4.4) and the statement that on the transverse hypersurface $\xi^+ = 0$ through the event $\gamma(\tau)$ Fermi coordinates are identical to Riemann coordinates around $\gamma(\tau)$. Moreover, as a Fermi expression, (4.5) is valid not only at a certain time τ but all along γ . Therefore its time derivative does not include new terms and one simply has

$$\partial_\tau X^A(\gamma, \xi) = \delta_+^A + \delta_a^A \partial_\tau \xi^{\bar{a}}. \quad (4.6)$$

as well as (evidently)

$$\partial_\sigma X^A(\gamma, \xi) = \delta_a^A \partial_\sigma \xi^{\bar{a}}. \quad (4.7)$$

Thus, provided that one imposes the lightcone gauge one can simultaneously use the attractive features of Riemann and Fermi coordinates, i.e. one can eat one's cake and have it too, and the covariant expansions of $X^A(\tau, \sigma)$ (4.5) and its derivatives (4.6, 4.7) become as simple as they could possibly be.

Moreover, by virtue of the identification (4.4), the transverse $\xi^+ = 0$ Riemann coordinate expansion (A.5) of the metric in terms of $\xi^{\bar{a}}$ is equivalent to the expansion (A.14) of the metric in Fermi coordinates.

Note that, in order to arrive at this conclusion, we only needed to impose the space-time diffeomorphism gauge condition that the metric be written in Fermi coordinates as well as the worldsheet diffeomorphism lightcone gauge condition $X^+ = \tau$. This is always possible.

Putting everything together, we conclude that in this combined lightcone (worldsheet) and Fermi (space-time) gauge, the expansion of the string e.o.m. around the null geodesic γ becomes identical, to all orders, actually term by term, to the lightcone gauged string theory e.o.m. in the Fermi coordinate expansion of the metric. Since the expansions agree term by term, this conclusion is valid both for the ordinary Fermi expansion (A.14) as well as for the Penrose-Fermi

expansion (A.18) of the metric (whose lowest order term is the Penrose limit plane wave) because the latter is in essence just a reordering of the former.

Frequently, the lightcone gauge is imposed in conjunction with the conformal gauge, and this imposes strong constraints on the background geometry which lead to the usual simplifications in the subsequent canonical quantisation. It is well known that the metrics for which the lightcone gauge can be imposed in addition to the conformal gauge are metrics of the Brinkmann form (B.1) admitting a parallel null vectorfield ∂_v [21]. Thus, if we insist on the conformal gauge (depending on the form of the metric, there may also be other suitable gauge choices leading to a tractable canonical formalism, see e.g. [26]), we need to understand for which Brinkmann metrics we can introduce Fermi coordinates compatible with the above Brinkmann form. In appendix B we establish the optimal result along these lines, namely that demanding the Fermi gauge, associated with any one of a spacetime filling congruence of null geodesics, imposes no further restrictions on the metric beyond those required by the lightcone and conformal gauge alone.

5 EXAMPLE: RIEMANN EXPANSION OF THE PLANE WAVE STRING EQUATIONS

To illustrate the above argument regarding the equivalence of the Riemann and Penrose-Fermi expansions, as a simple example we reconsider the plane wave in Brinkmann coordinates (3.1). These Brinkmann coordinates are Fermi coordinates for the central null geodesic $x^+ = \tau, x^{\bar{a}} = 0$, and the exact string e.o.m. and constraints, given in (3.2)-(3.5), are at most quadratic in the transverse fields $X^{\bar{a}}$. Their Riemann coordinate expansion, on the other hand, is *a priori* given by an infinite series. Thus our claim that these two expansions are (term by term) equivalent may at first appear to be puzzling.

To see what is going on, let us take a closer look at the second order Riemann coordinate string expansion of the e.o.m. (A.8) around the null geodesic. Using the rules (3.6), one finds

$$\begin{aligned} & (-\nabla_\tau^2 + \partial_\sigma^2)\xi^\lambda - R^\lambda_{\mu\rho_1\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu\xi^{\rho_1} \\ & - 2R^\lambda_{\rho_1\rho_2\mu}\dot{\gamma}^\mu\nabla_\tau\xi^{\rho_1}\xi^{\rho_2} - \frac{1}{2}[\nabla_{\rho_1}R^\lambda_{\mu\rho_2\nu} + \nabla_\mu R^\lambda_{\rho_1\rho_2\nu}]\dot{\gamma}^\mu\dot{\gamma}^\nu\xi^{\rho_1}\xi^{\rho_2} + \mathcal{O}((\xi)^3) = 0. \end{aligned} \quad (5.1)$$

Evaluating these in frame components, using the fact that for a plane wave the only nonvanishing component of the Riemann tensor is $R_{a+b+}(\tau)$, and after imposing the lightcone gauge $\xi^+ = 0$ (4.3), one finds the e.o.m.

$$\begin{aligned} & (\partial_\tau^2 - \partial_\sigma^2)(\xi^+ = 0) + \mathcal{O}((\xi)^3) = 0 \\ & (\partial_\tau^2 - \partial_\sigma^2)\xi^- + 2R_{+ab+}\partial_\tau\xi^a\xi^b + \frac{1}{2}\partial_+R_{+ab+}\xi^a\xi^b + \mathcal{O}((\xi)^3) = 0 \\ & (\partial_\tau^2 - \partial_\sigma^2)\xi^a + R^a_{+b+}\xi^b + \mathcal{O}((\xi)^3) = 0 \end{aligned} \quad (5.2)$$

and similarly the constraints

$$\begin{aligned} & \dot{\xi}^- - \frac{1}{2}R_{a+b+}\xi^a\xi^b + \frac{1}{2}\delta_{ab}(\dot{\xi}^a\dot{\xi}^b + \xi^{a'}\xi^{b'}) + \mathcal{O}((\xi)^3) = 0 \\ & \dot{\xi}^{-'} + \delta_{ab}\dot{\xi}^a\xi^{b'} + \mathcal{O}((\xi)^3) = 0. \end{aligned} \quad (5.3)$$

These equations are identical to the standard e.o.m. and constraints in Brinkmann/Fermi coordinates provided that all the higher order $\mathcal{O}((\xi)^{n \geq 3})$ terms in the Riemann expansion vanish. Thus the result of section 4 tells us that these terms have to be identically zero.

As a check on this geometric reasoning, in this case one can also establish the absence of these higher order terms in the Riemann coordinate expansion directly, by using some elementary combinatorial considerations similar to the kinds of arguments that are used to show [27] that plane wave (or pp-wave) backgrounds are exact solutions of string theory. Namely, as $\xi^+ = 0$, there are at most two contravariant $+$ indices, stemming from $\dot{\gamma} = E^+$. An initial R_{+a+b} contributes two covariant indices. Each additional power of the Riemann tensor adds another two covariant $+$ indices (since contractions are only possible over transverse indices), and each covariant derivative adds at least one, namely the $+$ -derivative (the others add two as can be seen by direct inspection of the Christoffel symbols). One covariant $+$ might be a free contravariant $-$ index (in the e.o.m. for ξ^-). Thus, denoting by r the number of Riemann tensors and by d the number of derivatives, we find the condition

$$2r + d - 1 \leq 2. \quad (5.4)$$

This implies that only terms with $r \leq 1$ and $d \leq 1$ can contribute, thus providing an alternative argument to the effect that the higher order terms in the expansion (5.2) are zero.

6 OUTLOOK

For practical applications, the key consequence of our work is the observation that in the combined Fermi/lightcone gauge, the naive expansion of the string coordinates (4.5) and their derivatives (4.6,4.7) is manifestly covariant. This should provide additional insight into, and significant simplification of, calculations performed e.g. in the AdS/CFT context (e.g. by extending the Fermi expansion of $AdS_5 \times S^5$ [15] to a string theory expansion).

Applications of this procedure are, however, not limited to the Penrose limit AdS/CFT context. For example, it was noted in [22] that the Penrose-Fermi expansion developed in [15], with γ interpreted as a photon trajectory, provides the ideal setting for performing certain QED calculations (like vacuum polarisation) in a curved background. It was also remarked there that it would be interesting to perform analogous calculations in string theory. We expect the formalism that we have developed in this paper, a stringy generalisation of [15], to be useful for that purpose.

The results obtained here also shed light on the propagation of strings in curved (and singular) backgrounds. For example, some of the observations in [18] regarding the string propagation through a big crunch / big bang singularity (namely that in the neighbourhood of such a cosmological singularity the string equations reduce to those in a plane wave) can be understood as a particular manifestation of the more general phenomenon that we have described here, since the plane wave in question is precisely the kind of singular homogeneous plane wave [11] that was shown in [13, 14] to arise generically as the Penrose limit of a space-time singularity.

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A TAYLOR EXPANSION IN RIEMANN AND FERMI COORDINATES

A.1 RIEMANN EXPANSION

The covariant expansion of a general space-time tensor using Riemann coordinates is discussed in detail in [17]. Here we can restrict ourselves to the embedding variables and the metric. First note that a coordinate difference $\Delta x^\mu = x^\mu - x_B^\mu$ of (nearby) points on the curved space-time manifold is an object whose transformation under space-time diffeomorphisms is not well defined. Thus a naive Taylor expansion in Δx^μ is bound to produce correct but nevertheless non-covariant equations. To circumvent this difficulty one can reparametrise $\Delta x^\mu(\xi)$ by a vector ξ sitting at x_B by means of the exponential map

$$x^\mu(x_B, \xi) = x_B^\mu + \Delta x^\mu(\xi) = (\text{Exp}_{x_B}(\xi))^\mu. \quad (\text{A.1})$$

As ξ^μ transforms as a vector, the ordinary Taylor expansion of the metric in terms of ξ^μ ,

$$G_{\mu\nu}(x_B + \Delta x(\xi)) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial \xi^{\rho_1}} \cdots \frac{\partial}{\partial \xi^{\rho_n}} G_{\mu\nu}(x_B) \xi^{\rho_1} \cdots \xi^{\rho_n}, \quad (\text{A.2})$$

has to be covariant, i.e. the coefficients can be re-expressed in terms of the curvature tensor and its covariant derivatives. Note, however, that in a general coordinate system the definition via the exponential map leads to a rather complicated dependence of $\Delta x(\xi)$ on ξ , namely

$$x^\mu(x_B, \xi) = x_B^\mu + \Delta x^\mu(\xi) = x_B^\mu + \xi^\mu - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_{\rho_1 \cdots \rho_n}^\mu \xi^{\rho_1} \cdots \xi^{\rho_n}, \quad (\text{A.3})$$

where $\Gamma_{\rho_1 \cdots \rho_n}^\mu = \nabla_{\rho_1} \cdots \nabla_{\rho_{n-2}} \Gamma_{\rho_{n-1} \rho_n}^\mu$ and ∇_ρ means covariant differentiation w.r.t. lower indices only. We see that in order to evaluate (A.2) one would also have to expand the coordinate functions x^μ themselves.

The solution to this problem is to promote x_B to be the origin of a new coordinate system ξ^μ in which geodesics emanating from x_B are straight lines. In these Riemann coordinates by definition one has $\Delta x^\mu = \xi^\mu$ or, equivalently,

$$x^\mu(x_B, \xi) = x_B^\mu + \xi^\mu, \quad (\text{A.4})$$

making them the natural choice of coordinate system to evaluate (A.2). Comparison of (A.3) and (A.4) shows that the symmetrised covariant derivatives of the Christoffel symbols vanish in Riemann coordinates, $\Gamma_{(\rho_1 \cdots \rho_n)}^\mu = 0$. From this relation one can iteratively express the partially symmetrised derivatives of the Christoffel symbols to arbitrary order in terms of the Riemann tensor, and then use these expressions to manifestly covariantise the expansion (A.2), leading to

$$G_{\mu\nu}(x_B + \xi) = G_{\mu\nu}(x_B) - \frac{1}{3} R_{\mu\rho_1\nu\rho_2} \xi^{\rho_1} \xi^{\rho_2} - \frac{1}{3!} \nabla_{\rho_1} R_{\mu\rho_1\nu\rho_2} \xi^{\rho_1} \xi^{\rho_2} \xi^{\rho_3} + \mathcal{O}((\xi)^4). \quad (\text{A.5})$$

As a tensorial equation, this is now valid in any coordinate system.

We also need to evaluate the derivative of the embedding variables X^μ , i.e. of the expansion (A.3). Here it is important to note that, while the symmetrised derivatives of the Christoffel symbols vanish in Riemann coordinates, this is not true for their ordinary derivatives. Therefore the derivative of (A.3) w.r.t. some parameter τ , e.g. along a curve in space-time, leads to an infinite series in Riemann coordinates,

$$\partial_\tau X^\mu(X_B, \xi) = \partial_\tau(X_B^\mu + \Delta X^\mu(\xi)) = \partial_\tau X_B^\mu + \partial_\tau \xi^\mu - \sum_{n=2}^{\infty} \frac{1}{n!} (\partial_\nu \Gamma_{\rho_1 \dots \rho_n}^\mu) \xi^{\rho_1} \dots \xi^{\rho_n} \partial_\tau X_B^\nu. \quad (\text{A.6})$$

In manifestly covariant form this reads

$$\begin{aligned} \partial_\tau X^\mu(X_B, \xi) &= \partial_\tau(X_B^\mu + \xi^\mu) = \partial_\tau X_B^\mu + \nabla_\tau \xi^\mu \\ &+ \left[-\frac{1}{3} R^\mu_{\rho_1 \nu \rho_2} \xi^{\rho_1} \xi^{\rho_2} + \frac{1}{12} \nabla_{\rho_1} R^\mu_{\rho_2 \rho_3 \nu} \xi^{\rho_1} \xi^{\rho_2} \xi^{\rho_3} \right] \partial_\tau X_B^\nu + \mathcal{O}((\xi)^4). \end{aligned} \quad (\text{A.7})$$

Putting everything together, we can now write down the expansion of the string e.o.m. (2.3),

$$\begin{aligned} \nabla_i \nabla^i \xi^\lambda + R^\lambda_{\mu \rho_1 \nu} \partial_i X_B^\mu \partial^i X_B^\nu \xi_1^\rho + 2R^\lambda_{\rho_1 \rho_2 \mu} \partial_i X_B^\mu \nabla^i \xi^{\rho_1} \xi^{\rho_2} \\ + \frac{1}{2} [\nabla_{\rho_1} R^\lambda_{\mu \rho_2 \nu} + \nabla_\mu R^\lambda_{\rho_1 \rho_2 \nu}] \partial_i X_B^\mu \partial^i X_B^\nu \xi^{\rho_1} \xi^{\rho_2} + \mathcal{O}((\xi)^3) = 0. \end{aligned} \quad (\text{A.8})$$

and of the constraints (2.4),

$$\begin{aligned} G_{\mu\nu} (2\nabla_\tau \xi^\mu \partial_\tau X_B^\nu + 2\nabla_\sigma \xi^\mu \partial_\sigma X_B^\nu + \nabla_\tau \xi^\mu \nabla_\tau \xi^\nu + \nabla_\sigma \xi^\mu \nabla_\sigma \xi^\nu) \\ - R_{\mu \rho_1 \nu \rho_2} \xi^{\rho_1} \xi^{\rho_2} (\partial_\tau X_B^\mu \partial_\tau X_B^\nu + \partial_\sigma X_B^\mu \partial_\sigma X_B^\nu) + \mathcal{O}((\xi)^3) = 0 \\ G_{\mu\nu} (\nabla_\tau \xi^\mu \partial_\sigma X_B^\nu + \nabla_\sigma \xi^\mu \partial_\tau X_B^\nu + \nabla_\tau \xi^\mu \nabla_\sigma \xi^\nu) - R_{\mu \rho_1 \nu \rho_2} \xi^{\rho_1} \xi^{\rho_2} \partial_\tau X_B^\mu \partial_\sigma X_B^\nu + \mathcal{O}((\xi)^3) = 0. \end{aligned} \quad (\text{A.9})$$

A.2 FERMION EXPANSION

Riemann coordinates are most suitable to evaluate covariant Taylor expansions around a point in space-time. However, if one wishes to expand only transversally to a given geodesic γ , i.e. a one-dimensional object, Fermi coordinates are the most adequate tool. In the following we will restrict the discussion to the case of null Fermi coordinates, i.e. with γ a null geodesic, considered in [15] and constructed as follows. First one introduces a quasi-orthonormal frame E_μ^A ,

$$ds^2|_\gamma = \eta_{AB} E^A E^B = 2E^+ E^- + \delta_{ab} E^a E^b \quad (\text{A.10})$$

parallel transported along γ , with $E_+^\mu = \dot{\gamma}^\mu$. The transversality condition is then implemented by $\xi_{\gamma(\tau)}^\mu E_\mu^+(\gamma(\tau)) = \xi_{\gamma(\tau)}^+ = 0$, where $\xi_{\gamma(\tau)}^\mu$ is the vector defining the Riemann coordinate system around the point $\gamma(\tau)$. The rôle of ξ^+ is now played by the affine parameter of the geodesic τ , promoted to be the Fermi coordinate $x^+ = \tau$. The remaining Fermi coordinates are identical to the Riemann coordinates restricted to the transverse hypersurface, i.e.

$$x^{\bar{a}} = E_{\mu}^{\bar{a}} \xi_{\gamma(\tau)}^\mu \Big|_{\xi^+=0} = \xi_{\gamma(\tau)}^{\bar{a}}. \quad (\text{A.11})$$

In Fermi coordinates, the Christoffel symbols as well as the symmetrised transverse components of their covariant or partial derivatives vanish all along γ ,

$$\Gamma_{AB}^C|_{\gamma} = \partial_{(\bar{a}_1} \dots \partial_{\bar{a}_{n-2}} \Gamma_{\bar{a}_{n-1}\bar{a}_n}^A|_{\gamma} = 0, \quad (\text{A.12})$$

and not only at a certain point, as for Riemann coordinates. The price we have to pay for this is that this is no longer true for the symmetrised higher derivatives including the geodesic direction (a lower $+$ -index). For example, while one obviously has $\Gamma_{BC,+}^A = 0$ by (A.12), one calculates e.g.

$$\Gamma_{(+B,C)}^A = R_{(BC)+}^A. \quad (\text{A.13})$$

Similarly to the Riemann case, the derivatives of the Christoffel symbols can be used to determine the explicit expansion of the metric in terms of the components of the Riemann tensor restricted to the geodesic γ . To cubic order (for the quartic terms see [15]) one finds

$$\begin{aligned} ds^2 = & 2dx^+ dx^- + \delta_{ab} dx^a dx^b \\ & - R_{+\bar{a}+\bar{b}} x^{\bar{a}} x^{\bar{b}} (dx^+)^2 - \frac{4}{3} R_{+\bar{b}\bar{a}\bar{c}} x^{\bar{b}} x^{\bar{c}} (dx^+ dx^{\bar{a}}) - \frac{1}{3} R_{\bar{a}\bar{c}\bar{b}\bar{d}} x^{\bar{c}} x^{\bar{d}} (dx^{\bar{a}} dx^{\bar{b}}) \\ & - \frac{1}{3} R_{+\bar{a}+\bar{b};\bar{c}} x^{\bar{a}} x^{\bar{b}} x^{\bar{c}} (dx^+)^2 - \frac{1}{4} R_{+\bar{b}\bar{a}\bar{c};\bar{d}} x^{\bar{b}} x^{\bar{c}} x^{\bar{d}} (dx^+ dx^{\bar{a}}) - \frac{1}{6} R_{\bar{a}\bar{c}\bar{b}\bar{d};\bar{e}} x^{\bar{c}} x^{\bar{d}} x^{\bar{e}} (dx^{\bar{a}} dx^{\bar{b}}) \\ & + \mathcal{O}(x^{\bar{a}} x^{\bar{b}} x^{\bar{c}} x^{\bar{d}}) \end{aligned} \quad (\text{A.14})$$

Turning now to the expansion of the coordinates and embedding variables, direct insertion of (A.13) into the expansion (A.3) leads to

$$x^A(\gamma, \xi) = \gamma^A + \Delta x^A(\xi) = \delta_+^A \tau + \xi^A - R_{+\bar{c}+}^A \xi^+ \xi^+ \xi^{\bar{c}} - 2R_{\bar{b}\bar{c}+}^A \xi^+ \xi^{\bar{b}} \xi^{\bar{c}} + \mathcal{O}((\xi)^3). \quad (\text{A.15})$$

In contrast to the Riemann expansion it contains terms of arbitrary high order in ξ^A (as long as $\xi^+ \neq 0$). However this expression is valid along γ . Accordingly we find, using (A.12), that no new terms appear after differentiation of the embedding variables,

$$\begin{aligned} \partial_{\tau} X^A(\gamma, \xi) &= \partial_{\tau}(\gamma^A + \Delta X^A(\xi)) \\ &= \delta_+^A + \partial_{\tau} \xi^A - \partial_{\tau}(R_{+\bar{c}+}^A \xi^+ \xi^+ \xi^{\bar{c}}) - 2\partial_{\tau}(R_{\bar{b}\bar{c}+}^A \xi^+ \xi^{\bar{b}} \xi^{\bar{c}}) + \mathcal{O}((\xi)^3) \end{aligned} \quad (\text{A.16})$$

A.3 PENROSE-FERMI EXPANSION

In [15] the Fermi expansion of the metric around a null geodesic was used to define a covariant extension of the Penrose limit to higher orders, i.e. a Penrose-Fermi expansion. In a nutshell the prescription is to rescale the Fermi coordinates together with a conformal transformation of the metric

$$(x_{\lambda}^+, x_{\lambda}^-, x_{\lambda}^a) = (x^+, \lambda^2 x^-, \lambda x^a), \quad ds_{\lambda}^2 = \frac{1}{\lambda^2} ds^2 \quad (\text{A.17})$$

This leads to a reshuffling of the terms in the Fermi expansion whose zero'th order term in λ is the Penrose limit plane wave associated with the metric $G_{\mu\nu}$ and the null geodesic γ ,

$$\begin{aligned} ds^2 = & 2dx^+ dx^- + \delta_{ab} dx^a dx^b - R_{a+b+} x^a x^b (dx^+)^2 \\ & + \lambda \left[-2R_{+a+-} x^a x^- (dx^+)^2 - \frac{4}{3} R_{+bac} x^b x^c (dx^+ dx^a) - \frac{1}{3} R_{+a+b;c} x^a x^b x^c (dx^+)^2 \right] + \mathcal{O}(\lambda^2) \end{aligned} \quad (\text{A.18})$$

B FERMI COORDINATES COMPATIBLE WITH THE BRINKMANN FORM

Here we want to show that there always exist Fermi coordinates (x^+, x^-, x^a) which are compatible with the general (Brinkmann) form

$$ds^2 = 2du(dv + A(u, y^k)du + A_i(u, y^k)dy^i) + G_{ij}(u, y^k)dy^i dy^j \quad (\text{B.1})$$

of a metric admitting a null parallel (and hence in particular Killing) vector ∂_v . This means that in this new coordinate system the metric has the same general form as above, and moreover has the features that (a) $x^+ = \tau$, $x^- = 0$, $x^a = 0$ is the basic null geodesic γ , (b) $\partial_+|_\gamma, \partial_-|_\gamma, \partial_a|_\gamma$ is a quasi-orthonormal parallel frame along γ , and (c) all the curves $x^+ = c^+$, $x^- = c^-t$, $x^a = c^a t$ with c^+ , c^- , $c^a = \text{const.}$ are also geodesics.

In order to identify a suitable null geodesic γ (actually, as we will see, a whole congruence of null geodesics), we first cast the Brinkmann metric (B.1) into the Rosen coordinate form

$$ds^2 = 2dudv + G_{ij}(u, y^k)dy^i dy^j, \quad (\text{B.2})$$

which is always possible [27]. It is now readily checked that any curve $u = p^v \tau$, $p^v \neq 0$, with $v, y^i = \text{const.}$ is a null geodesic. Pick one of this congruence, set $p^v = 1$, call it γ , shift v so that γ sits at $(v = 0, y^i = y_0^i)$, and introduce the corresponding Fermi coordinate $x^+ = u = \tau$.

Moreover, $x^+ = c^+$ ($p^v = 0$) is also a solution to the geodesic e.o.m. and thus the hypersurfaces $x^+ = c^+$ can be generated by transverse geodesics emanating from the intersection point with γ . ∂_v is parallel and hence, in particular, parallel transported along γ . Choose $E_+ = \dot{\gamma}$ and $E_- = \partial_v$ and complete it by $E_a = E_a^i \partial_i$ to a quasi-orthonormal parallel frame along γ . In any one of the spacelike codimension 2 surfaces $v, x^+ = \text{const.}$ spanned by the y^i , with induced metric $G_{ij}(x^+, y^k)$, we introduce Riemann normal coordinates x^a around the point (y_0^i) w.r.t. the frame $E_a(x^+)$, i.e. such that $\partial_a|_\gamma = E_a$. Since $G_{ij}(x^+, y^k)$ is independent of v , this can be achieved by a v -independent, but generically x^+ -dependent, coordinate transformation of the form $x^a = x^a(x^+, y^i)$. Then the metric takes the form

$$ds^2 = 2dx^+(dv + A(x^+, x^c)dx^+ + A_a(x^+, x^c)dx^a) + G_{ab}(x^+, x^c)dx^a dx^b. \quad (\text{B.3})$$

Note that, while this has the same general form as (B.1), the coordinates are now such that (a) $x^+ = \tau$, $v = 0$, $x^a = 0$ is the Fermi null geodesic γ , and (b) $\partial_+, \partial_v, \partial_a$ is parallel quasi-orthonormal along γ . Furthermore, the geodesic e.o.m. for the x^a are satisfied by $x^a = c^a t$ with $x^+ = c^+$, since A and A_a do not contribute for $\dot{x}^+ = 0$ and the x^a are spatial Riemann coordinates for G_{ab} .

To completely satisfy criterion (c), we still need to replace v by a coordinate x^- whose geodesic e.o.m. are fulfilled by $x^- = c^-t$, $x^a = c^a t$ and $x^+ = c^+$ for all c^+, c^-, c^a , and such that ∂_- is quasi-orthonormal parallel along γ . The only coordinate transformation left to us is a shift $x^- = v + P(x^+, x^a)$. Note that this shift changes only A and A_a in (B.3) and therefore does not effect the e.o.m. for x^a if $\dot{x}^+ = 0$. Furthermore, if P is at least quadratic in the x^a , the Jacobian of the coordinate transformation is trivial on γ , and therefore $\partial_+, \partial_-, \partial_a$ is parallel along γ because there it is identical to the above parallel frame $\partial_+, \partial_v, \partial_a$.

After the shift, the x^- e.o.m. is

$$\begin{aligned} 2\ddot{x}^- &= -\partial_a \partial_b P(c^+, c^d t) c^a c^b - \frac{d}{dt} (A_a(c^+, c^d t) c^a + \partial_{x^+} G_{ab}(c^+, c^d t) c^a c^b) \\ &= -\partial_a \partial_b P(c^+, c^d t) c^a c^b - \partial_a (A_b(c^+, c^d t) c^a c^b + \partial_{x^+} G_{ab}(c^+, c^d t) c^a c^b) \end{aligned} \quad (\text{B.4})$$

where we used $\dot{x}^+ = 0$. We want the right side to vanish. Rescaling c^a by t we get

$$\partial_a \partial_b P(c^+, c^d) c^a c^b = -\partial_a A_b(c^+, c^d) c^a c^b + \partial_{x^+} G_{ab}(c^+, c^d) c^a c^b \equiv D_{ab}(c^d) c^a c^b. \quad (\text{B.5})$$

Expanding both sides in a Taylor series in the c^a , comparison of coefficients gives

$$\partial_{(a_1} \cdots \partial_{a_n)} P(c^+, 0) = \partial_{(a_1} \cdots \partial_{a_{n-2}} D_{a_{n-1} a_n)}(c^+, 0). \quad (\text{B.6})$$

This can always uniquely be solved for given D_{ab} . Finally, as A_a is at least linear in the x^a (the metric restricted to γ is flat) and $\partial_{x^+} G_{ab}$ is at least quadratic (Riemann coordinate metric), P is also at least quadratic in the x^a , as required.

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